

Wednesday, Oct 8, Lecture 17: Cauchy-Riemann, contour integration

Last time...

• $f: \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic if

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

• f is holo $\Leftrightarrow f$ is real-diff + "Extra structure"

• Ex: $f(z) = |z|^2$ is real-differentiable, since

$$f(x+iy) = x^2 + y^2,$$

but not holomorphic.

— Usual properties of derivative hold for holomorphic functions: $a, b \in \mathbb{C}$ and $f, g: \mathbb{C} \rightarrow \mathbb{C}$

a) (Linearity) If $a, b \in \mathbb{C}$, then

$$(af + bg)'(z) = af'(z) + bg'(z)$$

b) (Leibniz Rule)

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

c) (Chain Rule)

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Cauchy-Riemann Equations

Any real-differentiable $f: \mathbb{C} \rightarrow \mathbb{C}$ has a Jacobian.
write $u: \mathbb{R}^2 \rightarrow \mathbb{R}$ $v: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$f(x+iy) = u(x,y) + i v(x,y)$$

$$Df(z_0) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \bigg|_{z_0}$$

For f to be holomorphic, need the limits from all directions to be equal:

$$f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$$

• Let's first take $h \in \mathbb{R}$. Let $z = x+iy$.

$$\frac{f(z+h) - f(z)}{h} = \frac{u(x+h,y) + i v(x+h,y) - u(x,y) - i v(x,y)}{h}$$

$$= \frac{u(x+h,y) - u(x,y)}{h} + i \frac{v(x+h,y) - v(x,y)}{h}$$

$$\lim_{h \rightarrow 0} \rightarrow = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

• Now, taking $ih \in i\mathbb{R}$,

$$\begin{aligned} \frac{f(z+ih) - f(z)}{ih} &= \frac{u(x, y+h) + i v(x, y+h) - u(x, y) - i v(x, y)}{ih} \\ &= \frac{1}{i} \frac{u(x, y+h) - u(x, y)}{h} + \frac{i}{i} \left(\frac{v(x, y+h) - v(x, y)}{h} \right) \\ \lim_{h \rightarrow 0} &\rightarrow \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned}$$

Equating limits,

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \quad (*)$$

Cauchy Riemann Equations!

This is a condition on the Jacobian:

if $\frac{\partial u}{\partial x}(z_0) = a$ and $\frac{\partial v}{\partial x}(z_0) = b$,

$$Df(z_0) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

HW! This is the Jacobian of multiplication by a complex number $a+bi$

Thm:

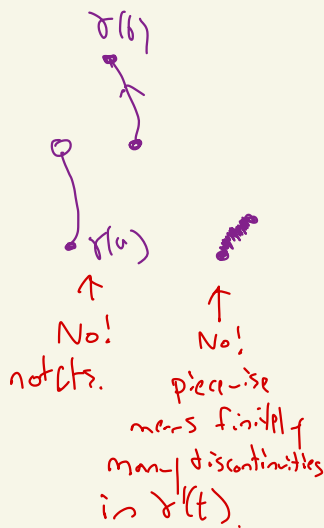
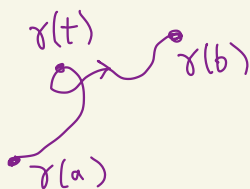
f is holomorphic $\Leftrightarrow f(x,y) = u(x,y) + i v(x,y)$ is real-differentiable and satisfies the CR-equations $(*)$.

Contour Integration

These are exactly line integrals from Calc III.

We will work on ^{oriented} continuous, piecewise smooth curves

$$\gamma: [a, b] \rightarrow \mathbb{C}:$$



Orientation = it has a direction!

Recall: chain rule

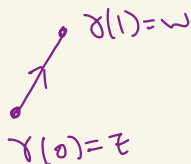
$$f: \mathbb{C} \rightarrow \mathbb{C}, \gamma: [a, b] \rightarrow \mathbb{C}.$$

$$(f \circ \gamma)'(t) = f'(\gamma(t)) \cdot \gamma'(t)$$

Two common paths:

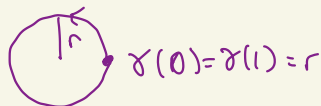
• (Lines) $\gamma: [0, 1] \rightarrow \mathbb{C}$, fix z, w .

$$\gamma(t) = tw + (1-t)z$$



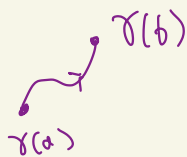
• (circles) $\gamma: [0, 1] \rightarrow \mathbb{C}$, fix radius r .

$$\gamma(t) = re^{it2\pi}$$



In vector calc, we integrate $u: \mathbb{R}^2 \rightarrow \mathbb{R}$ along contours $\gamma: [a, b] \rightarrow \mathbb{R}^2$ by

$$\int_{\gamma} u = \int_a^b u(\gamma(t)) \gamma'(t) dt.$$



So now:

$$\begin{aligned} \int_{\gamma} f(x+iy) &= \int_{\gamma} u + i \int_{\gamma} v = \int_a^b u(\gamma(t)) \gamma'(t) dt + i \int_a^b v(\gamma(t)) \gamma'(t) dt \\ &= \int_a^b f(\gamma(t)) \gamma'(t) dt \end{aligned}$$

Def:

$f: \mathbb{C} \rightarrow \mathbb{C}$ and $\gamma: [a, b] \rightarrow \Omega$ is a smooth curve.

Then the contour integral of f along γ is

$$\int_{\gamma} f = \int_{\gamma} f(z) dz := \int_a^b f(\gamma(t)) \gamma'(t) dt$$

• Note that curves come with an orientation:



So

$$\int_{-\gamma} f(z) dz = - \int_{\gamma} f(z) dz$$

Special case:

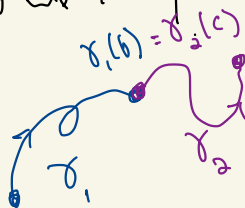
$f: \mathbb{R} \rightarrow \mathbb{R}$,

$$\int_b^a f(t) dt = - \int_a^b f(t) dt$$



And we may concatenate curves, as long as

$$\begin{aligned} \gamma_1: [a, b] &\rightarrow \mathbb{C} \\ \gamma_2: [c, d] &\rightarrow \mathbb{C} \\ \gamma_1(b) &= \gamma_2(c) \end{aligned}$$



$$\Rightarrow \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz = \int_{\gamma_1 + \gamma_2} f(z) dz$$

We want integrals to depend only on the curve $\gamma([a,b])$,
not the parametrization.

Eg,



$\gamma \quad e^{izt}, t \in [0, \pi)$ should be the same as
 $\tilde{\gamma} \quad e^{it}, t \in [0, 2\pi)$

$$\int_{\gamma} f dz = \int_0^{\pi} f(e^{izt}) \cdot 2i e^{izt} dt$$

$$\int_{\tilde{\gamma}} f dz = \int_0^{2\pi} f(e^{it}) \cdot i e^{it} d\tilde{t}$$

related b-1
 $\tilde{t} = 2t$

in general,
 $\tilde{t} = \alpha(t)$
 α differentiable
 and orientation
 preserving.

Prop: (reparametrization)

If $\tilde{\gamma}$ a reparam. of γ ,

$$\int_{\tilde{\gamma}} f = \int_{\gamma} f$$

for any cts. f .

Pf:

By def.

$$\begin{aligned} \int_{\gamma} f &= \int_a^b f(\gamma(t)) \cdot \gamma'(t) dt \\ &= \int_a^b f(\tilde{\gamma}(\alpha(t)) \cdot \tilde{\gamma}'(\alpha(t)) \cdot \alpha'(t) dt \\ &= \int_{\tilde{a}}^{\tilde{b}} f(\tilde{\gamma}(s)) \tilde{\gamma}'(s) ds = \int_{\tilde{\gamma}} f \end{aligned}$$

chain rules:
 $\gamma'(t) = \tilde{\gamma}'(\alpha(t)) \cdot \alpha'(t)$

$$\frac{ds}{dt} = \frac{ds}{dt}$$

$$\alpha(t) = s, \quad \alpha(a) = \tilde{a}, \quad \alpha(b) = \tilde{b} \rightarrow$$

Moral:
 $\int_{\gamma} f$ depends on contour,
~~not~~ γ
 not parametrization

